MATH 4573: HOMEWORK 8

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Due: April 5, 2024.

This homework has two sections: the first section has the problems that you'll turn in for credit. The second section contains recommended problems from the textbook, myself or other sources; you are not required to do these, but I recommend that you check them out.

For any problem in this assignment, **you must show all of your work in order to receive full credit.** Please do not use words such as "clear", "obvious" or "trivial" in your solutions.

Your solutions should not use theorems from sections which come after the day the homework was assigned. This HW should use up to what we've covered in class so far (including "§5.0", §5.1 and §5.3).

1. Problems to Submit

Exercise 1. Determine whether the following Diophantine equations have integral solutions. If they do, give a complete description of their solutions (don't forget to show your work for calculating GCD's of larger numbers).

a) $10x - 7y = 17$.
b) $903x + 731y = 60$.
c) $mx + (m+1)y = 10$, where $m > 1$ is a fixed positive integer.
d) $(n-1)x + (n+1)y = 4573$, where $n > 1$ is a fixed odd integer.

Exercise 2.
Show that the line

$$L : ax + by = c$$

has an integral point if and only if for any integer $n \in \mathbb{Z}$, the line

$$L_n : ax + by = na + c$$

has an integral point. Briefly argue that this still holds if we replace na with nb in L_n.

Exercise 3. Show that the Diophantine equation

$$x^2 + y^2 = 9z + 6$$

has no integral solutions.

Exercise 4. Show that the Diophantine equation

$$x^8 + 1 = 7y$$

has no integral solutions. However, demonstrate that it has infinitely many rational solutions.

Exercise 5. Show that every Pythagorean triple (x, y, z) is such that 3 divides (at least) one of x, y, z and 5 divides (at least) one of x, y, z.

Exercise 6. Determine all primes $p \in \mathbb{Z}^+$ such that the equation

$$x^2 - y^2 = p$$

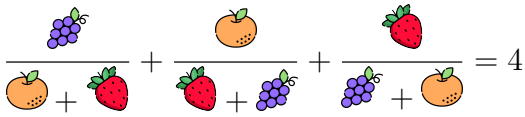
has integral solutions.

Exercise 7. Who did you consult for this assignment? What resources did you use?

2. Other Recommended Problems

From the textbook, pages $218 - 219$: $\#2 - 7, 10 - 12$.
Page 213: $\#1, 4 - 8$.

Bonus Exercise 8. $> 99\%$ of people can't solve this! Find all  ,  ,  $\in \mathbb{Z}^+$ with



(see e.g. https://mathoverflow.net/questions/227713/estimating-the-size-of-solutions-of-a-diophantine-equation).

Bonus Exercise 9. $99.\overline{9}\%$ of people cannot solve this one! For  $\in \mathbb{Z}$ with  ≥ 3, find all  ,  ,  $\in \mathbb{Z}^+$ with

References

[NZM91] I. Niven, H.S. Zuckerman and H.L. Montgomery, *An introduction to the theory of numbers*, 5th Ed., John Wiley & Sons, Inc., New York (1991).